Jianxiong Wang

The maximum principle

Thm: Consider $L = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i}$, $\Omega \subseteq \mathbb{R}^n$ open. s.c. bdd.

If $Lu = 0$ in Ω, (a_{ij}) is p.d. and u attains its maximum over $\bar{\Omega}$ at an interior pt. then $u = 0$.

Rmk: Uniform ellipticity of L $\Rightarrow$ positive definiteness of (a_{ij})

MSE: $$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

$$= (1+u_y^2) u_{xx} + (1+u_x^2) u_{yy} - 2u_x u_y u_{xy}$$

main result

Thm (Cor 1.27): Let u_1, u_2 be two solutions to MSE. sps Ω is connected. $u_1 \le u_2$ and $u_1(p) = u_2(p)$ for some $p \in \Omega$, then $u_1 \equiv u_2$.

firstly. given two solution's u_1, u_2, , $v = u_2 - u_1$ satisfies a unif. elliptic divergence form equation.

Lemma 1.26 If u_1, u_2 are solutions of the MSE on $\Omega \subseteq \mathbb{R}^n$, then $v = u_2 - u_1$ satisfies

$$\operatorname{div}(a_{ij} \nabla v) = 0,$$

where the eigenvalues of matrix $a_{ij} = a_{ij}(x)$ satisfy

$$0 < \mu < \lambda_1 \leq \cdots \leq \lambda_n \leq \frac{1}{\mu}.$$

μ is a constant depending on the upper bounds for $|\nabla u_i|$.

Proof: By MSE, we have

$$\operatorname{div} \frac{\nabla u_1}{\sqrt{1+|\nabla u_1|^2}} = \operatorname{div} \frac{\nabla u_2}{\sqrt{1+|\nabla u_2|^2}} = 0$$

define $F: \mathbb{R}^n \to \mathbb{R}^n$ by $F(a) = \dfrac{a}{\sqrt{1+|a|^2}}$, then

$$\operatorname{div}(F(\nabla u_1) - F(\nabla u_2)) = 0$$

$$F(\nabla u_2) - F(\nabla u_1) = \int_0^1 \frac{d}{dt}\left(F(\nabla u_1 + t(\nabla u_2 - \nabla u_1))\right) dt$$

$$= \int_0^1 dF(\nabla u_1 + t(\nabla u_2 - \nabla u_1)) \cdot \nabla(u_2 - u_1)\, dt$$

$$= \int_0^1 dF(\nabla u_1 + t(\nabla u_2 - \nabla u_1))\, dt \cdot \underbrace{\nabla(u_2 - u_1)}_{\nabla v}$$

then $\operatorname{div}(a_{ij} \nabla v) = 0$.

For $v \in S^{n-1}$, $x \in \mathbb{R}^n$, $dF_x(v) = \dfrac{v}{(1+|x|^2)^{1/2}} - \dfrac{\langle x, v\rangle x}{(1+|x|^2)^{3/2}}$

$$\langle dF_x(v), v\rangle = \frac{|v|^2}{(1+|x|^2)^{1/2}} - \frac{\langle x, v\rangle^2}{(1+|x|^2)^{3/2}} > 0 \quad \text{if } v \neq 0.$$

So (a_{ij}) is positive definite. $\square$

Proof of the thm:

by max principle $Lv = 0$, and $v(p) = 0$, then $v \equiv 0$. $\square$

Corollary: If $\Sigma_1, \Sigma_2 \subset \mathbb{R}^n$ are complete connected minimal hypersurfaces s.t. Σ_2 lies on one side of Σ_1, then $\Sigma_1 = \Sigma_2$.

(also true for $\Sigma \subset (M, g)$)

Rado - Schoen Theorem $\longrightarrow$ moving plane method

First variation formula

$$\frac{d}{dt}\Big|_{t=0} \mathrm{vol}(F(\Sigma, t)) = -\int_\Sigma \langle F_t, H\rangle = \int_\Sigma \mathrm{div}_\Sigma F_t.$$

Σ is critical pt of area iff $H = 0$.

Second variation formula

Consider the variation $F: \Sigma^k \times (-\varepsilon, \varepsilon) \longrightarrow M^{n+1}$ satisfying

(1) $\Sigma^k \subset M^n$ is minimal with trivial normal bundle

(2) $F(\cdot, 0) = Id$

(3) F_t has compact supp.

(4) $F_t^T \equiv 0$ it is a normal variation

Then.
$$\frac{d^2}{dt^2}\Big|_{t=0} vol(F(\Sigma, t)) = -\int_\Sigma \langle F_t, L F_t \rangle$$

L is called the stability operator

$$L = \Delta^N_\Sigma + Ric_M(n, n) + \tilde{A}$$

where $\tilde{A}$ is Simon's operator by

$$\tilde{A}(X) = \sum_{i,j=1}^k g(A(E_i, E_j), X)\, A(E_i, E_j) \qquad A(X, Y) = (\nabla_X Y)^N$$

$$\Delta^N_\Sigma = \sum_{i=1}^k (\nabla_{E_i} \nabla_{E_i} X)^N - \sum_{i=1}^k (\nabla_{(\nabla_{E_i} E_i)^T} X)^N$$

Proof:

let x_i be local coordinates on Σ, and orthonormal at x

Set

$$g_{ij}(t) = g(F_{x_i}, F_{x_j}), \quad \nu(t) = \sqrt{\det(g_{ij}(t))}\sqrt{\det g^{ij}(0)}$$

$$\nu'(t) = \frac{1}{2} tr(g_{ij}'(t) g^{lm}(t))\, \nu(t) \qquad \nu' = \frac{1}{2} tr(g^{-1} g')\, \nu$$

differentiating $\nu'(t)$ yields

$$\nu''(t) = \frac{1}{2}\nu'(t)\, tr(g^{-1}g') + \frac{1}{2}\nu(t)\, tr(g''g^{-1}) + \frac{1}{2}\nu(t)\, tr(g'(g^{-1})')$$

Since $gg^{-1} = I$, $(g^{-1})' = -g^{-1}g'g^{-1}$

at $t=0$, $\nu=1$, $\nu'=0$, $g_{ij}=\delta_{ij}$

$$\nu''(t)\big|_{t=0} = 0 + \frac{1}{2} tr(g''(0)) - \frac{1}{2} tr(g'(0)g'(0))$$

$$= \underline{\frac{1}{2} tr(g''(0)) - \frac{1}{2}|g'(0)|^2} \quad \text{since } tr(A^2) = |A|^2 \text{ if } A = A^T.$$

Now compute these two terms :

Lemma: ① $|g'(0)|^2 = 4|\langle A(\cdot,\cdot), F_t\rangle|^2$

② $tr(g''(0)) = 2|\langle A(\cdot,\cdot), F_t\rangle|^2 + 2|\nabla^N_\Sigma F_t|^2 + 2Ric(F_t, F_t)$

$+ 2\, div_\Sigma(F_{tt})$

proof: ① $g_{ij} = \langle F_{x_i}, F_{x_j}\rangle$

$$\Rightarrow g'_{ij} = \langle F_{tx_i}, F_{x_j}\rangle + \langle F_{x_i}, F_{tx_j}\rangle$$

$$= -\langle F_t, \nabla^N_{F_{x_i}} F_{x_j}\rangle - \langle F_t, \nabla^N_{F_{x_i}} F_{x_j}\rangle$$

$$= -2\langle A(F_{x_i}, F_{x_j}), F_t\rangle$$

② $tr(g''(0)) = 2\sum_{i=1}^{k}\langle F_{x_itt}, F_{x_i}\rangle + 2\sum_{i=1}^{k}\langle F_{x_it}, F_{x_it}\rangle$

first term:

$$\sum_{i=1}^{k}\langle F_{x_itt}, F_{x_i}\rangle = \sum_{i=1}^{k}\langle \nabla_{F_t}\nabla_{F_t} F_{x_i}, F_{x_i}\rangle = \sum_{i=1}^{k}\langle \nabla_{F_t}\nabla_{F_{x_i}} F_t, F_{x_i}\rangle$$

$$= \sum_{i=1}^{k}\langle R(F_{x_i}, F_t)F_t, F_{x_i}\rangle + \sum_{i=1}^{k}\langle \nabla_{F_{x_i}}\nabla_{F_t} F_t, F_{x_i}\rangle$$

$[F_{x_i}, F_t] = 0$

$$= \sum_{i=1}^{k}\langle R(F_{x_i}, F_t)F_t, F_{x_i}\rangle + div_\Sigma F_{tt}$$

$R(u,v)w = \nabla_v\nabla_u w - \nabla_u\nabla_v w + \nabla_{[u,v]}w$

$$\operatorname{tr} g''(0) = 2\sum_{i=1}^{k} \langle R(F_{x_i}, F_t) F_t, F_{x_i} \rangle + 2\,\operatorname{div}_\Sigma F_{tt}$$

$$+ 2\sum_{i=1}^{k} \langle F_{x_i t}^{T}, F_{x_i t}^{T} \rangle + 2\sum_{i=1}^{k} \langle F_{x_i t}^{N}, F_{x_i t}^{N} \rangle$$

$$= 2\,\underbrace{\operatorname{tr}_\Sigma \langle R(\cdot, F_t) F_t, \cdot \rangle}_{= Ric(F_t, F_t)} + 2\operatorname{div}_\Sigma F_{tt}$$

$$+ 2|\langle A(\cdot,\cdot), F_t \rangle|^2 + 2|\nabla_\Sigma^N F_t|^2 \qquad \square$$

Therefore. $\gamma''(t) = -|\langle A(\cdot,\cdot), F_t \rangle|^2 + |\nabla_\Sigma^N F_t|^2 - Ric(F_t, F_t) + \operatorname{div}_\Sigma(F_{tt})$

Besides, $\frac{d^2}{dt^2}\Big|_{t=0} vol(F(\Sigma, t)) = \int_\Sigma \frac{d^2}{dt^2}\Big|_{t=0} \gamma(t)\sqrt{\det g_{ij}(0)}$

then $= -\int_\Sigma |\langle A(\cdot,\cdot), F_t|^2 + \int_\Sigma |\nabla_\Sigma^N F_t|^2 - \int_\Sigma Ric(F_t, F_t)$

$= -\int_\Sigma \langle F_t, L F_t \rangle$. $\square$

Ref: A course in minimal surfaces — Colding and Minicozzi

Minimal surface — Sun